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DIET | DIPARTIMENTO DI INGEGNERIA DELL'INFORMAZIONE, ELETTRONICA E TELECOMUNICAZIONI

Uso delle trasformate bidimensionali di Hermite e Laguerre nell'elaborazione delle immagini e dei segnali provenienti da array di sensori

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Abstract

- Hermite and Laguerre polynomials costitute a well known interchangeable basis for MOM and differential equation solutions:
  - Harmonic quantum oscillator (Schrödinger equation with quadratic potential);
  - Gaussian quadrature formulas;
  - Lossy transmission lines;
  - Gram-Charlier and Edgeworth expansion in statistics.
- Application of Hermite and Laguerre expansions to sensor array modeling and image and multichannel signal processing exhibits a great flexibility and power due to an amazing set of invariance and transformation properties.

1-D Hermite-Gauss functions

- The product of Hermite polynomials by a Gaussian window function allows to build a complete and orthogonal set of functions in the interval (-∞,+∞).

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n\left(e^{-x^2}\right)}{dx^n}; \quad H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

$$\int_{-\infty}^{+\infty} H_m(x) H_n(x) e^{-x^2} dx = \sqrt{\pi}\, 2^n n! \delta_{nm}$$

$$h_n(x;\sigma) = \frac{H_n\left(\frac{x}{\sigma}\right)}{\sqrt{2^n n! \sigma \sqrt{\pi}}} e^{-\frac{x^2}{2\sigma^2}}$$

Physicist's Hermite polynomial

Three-term recurrence

Orthogonality condition

Hermite-Gauss function of scale σ

1-D Hermite-Gauss expansion

- Taylor-type convergence: approximation breaks down beyond a critical distance from origin;
- Isomorphic w.r.t. Fourier transform;
- Compact support in both time (space) and frequency.

1-D Hermite expansion

$$x(t) = \sum_{n=0}^{\infty} b_n h_n(t;\sigma)$$

$$\int_{-\infty}^{+\infty} \left[H_m(x) e^{-\frac{x^2}{2}} \right] e^{-j\omega x} dx = \left[(-j)^m \sqrt{2\pi} \right] \left[H_m(\omega) e^{-\frac{\omega^2}{2}} \right]$$

Fourier eigenvalue

2-D Hermite-Gauss (2-D HG) expansion

- More advantages are obtained by a 2-D Hermite expansion of images (space/space) and array signals (space/time or space/space).

2-D HG Expansion

$$I(x,t)=\sum_{m=0}^{\infty}\sum_{l=0}^{m} f_{m-l,l}\,\phi_{m-l,l}\left(x,t;\sigma\right)$$

$$\phi_{m-l,l}\left(x,t;\sigma\right)=\frac{H_{m-l}\left(x/\sigma\right)}{\sqrt{2^{m-l}\left(m-l\right)!\sigma\sqrt{\pi}}}\cdot\frac{H_{l}\left(t/\sigma\right)}{\sqrt{2^{l}\,l!\sigma\sqrt{\pi}}}\,e^{-\frac{x^2+t^2}{2\sigma^2}}$$

2-D HG basis function

Laguerre-Gauss 2-D expansion

$$I(\rho,\gamma) = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} l_{n,k} \mathcal{L}_{n,k} (\rho,\gamma;\sigma)$$

Polar 2-D Laguerre expansion

$$\rho = \sqrt{q^2 + r^2}; \quad \gamma = \arctan(r/q)$$

$$\mathcal{L}_k^{(n)}(\rho,\gamma;\sigma) = \frac{1}{\sqrt{k!(|n|+k)!}} (-1)^k \left(\frac{\rho}{\sigma}\right)^{|n|} L_k^{(|n|)}\left[\left(\frac{\rho}{\sigma}\right)^2\right] \frac{1}{\sigma\sqrt{\pi}} e^{-\frac{1}{2}\left(\frac{\rho}{\sigma}\right)^2} e^{jn\gamma}$$

$$L_k^{(n)}(x) = \sum_{p=0}^{k} (-1)^p \binom{n+k}{k-p} \frac{x^p}{p!}$$

Generalized Laguerre polynomial

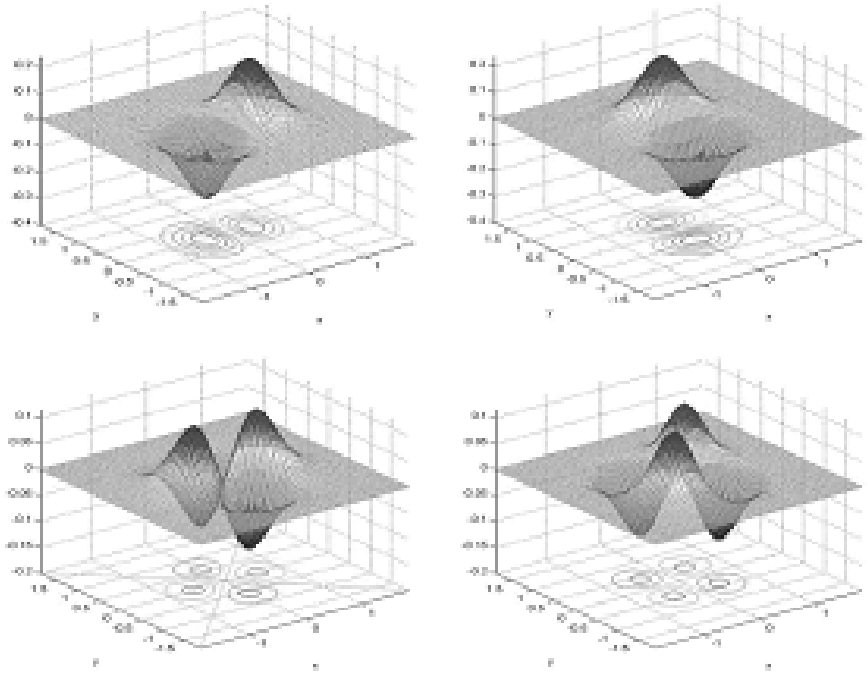
Laguerre-Gauss (LG-CH) function properties

- Characterized by *scale* σ, *radial order k* and *angular order n* : circular harmonics.
- Rotate by multiplication with a phase factor.
- Compact 2-D circular support region.

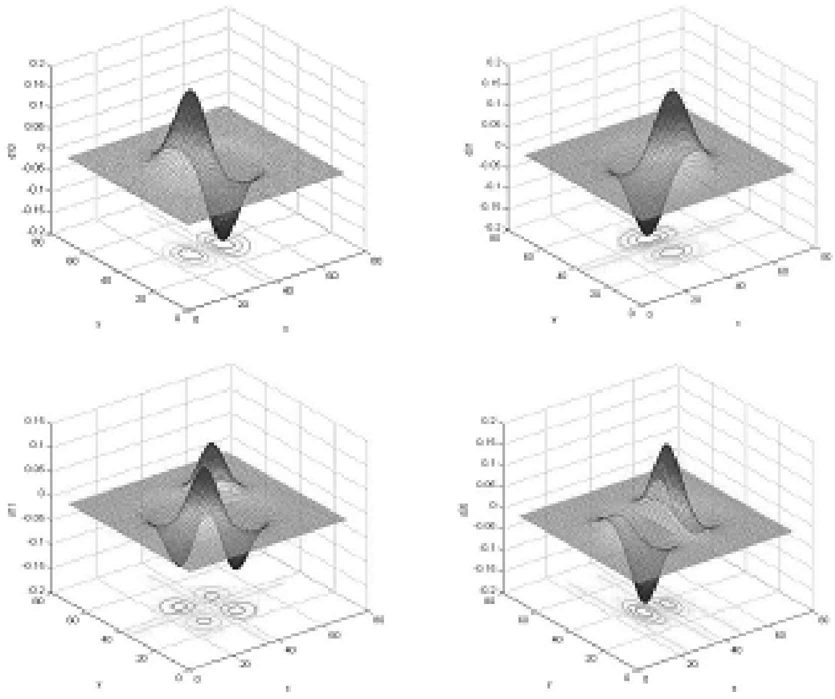
$$\rho < \sigma\sqrt{4M+1}; \quad |\omega| < \frac{\sqrt{4M+1}}{\sigma}$$

LG-CH and 2-D HG function shapes

- LG-CH functions



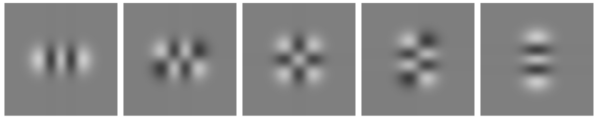
- 2-D HG functions



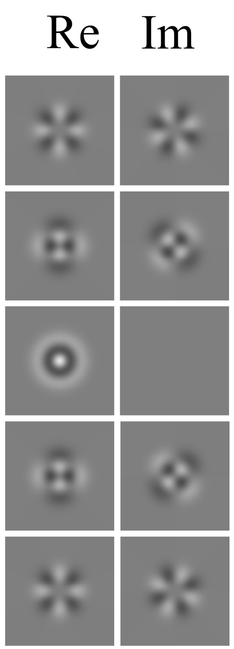
Block transformation property

- 2-D HG and LG-CH functions and expansion coefficients are mutually related by an unitary, block-diagonal transformation:

$$\mathbf{g}_M = \begin{bmatrix} \mathbf{g}_0 \\ \mathbf{g}_1 \\ \vdots \\ \mathbf{g}_M \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{T}_M \end{bmatrix} \begin{bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_M \end{bmatrix} = \mathbf{T}\mathbf{f}_M$$

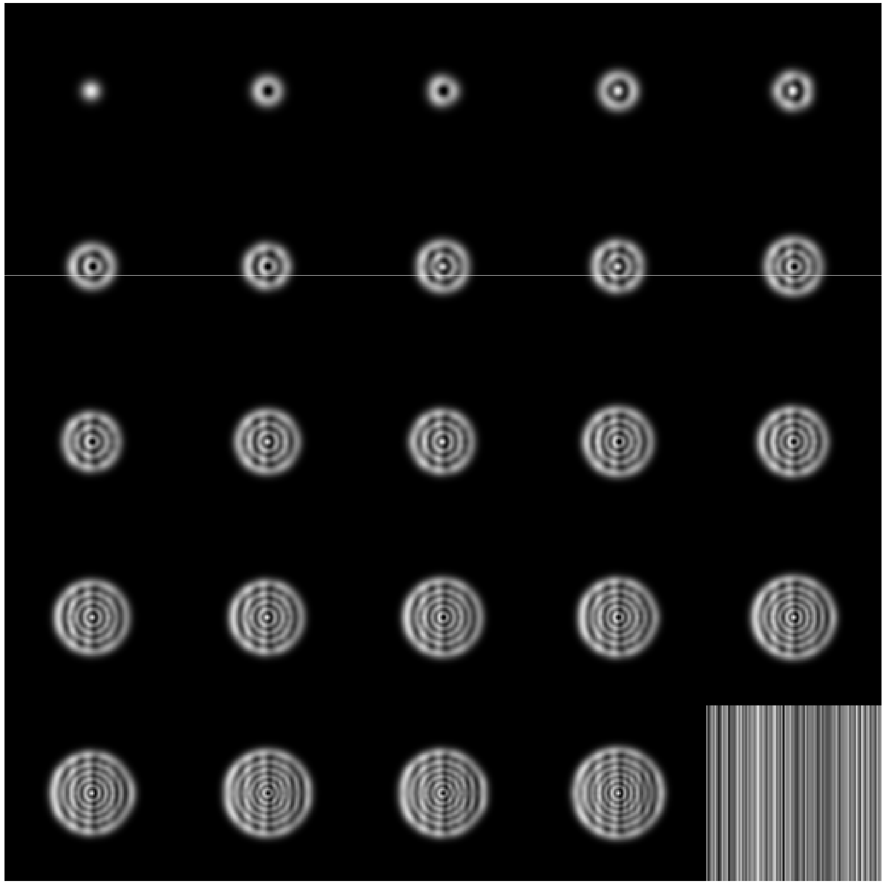


$\frac{1}{4}$	$-\frac{j}{2}$	$-\frac{\sqrt{6}}{4}$	$\frac{j}{2}$	$\frac{1}{4}$
$\frac{1}{2}$	$-\frac{j}{2}$	0	$-\frac{j}{2}$	$-\frac{1}{2}$
$\frac{\sqrt{6}}{4}$	0	$\frac{1}{2}$	0	$\frac{\sqrt{6}}{4}$
$\frac{1}{2}$	$\frac{j}{2}$	0	$\frac{j}{2}$	$-\frac{1}{2}$
$\frac{1}{4}$	$\frac{j}{2}$	$-\frac{\sqrt{6}}{4}$	$-\frac{j}{2}$	$\frac{1}{4}$



Locality property

- 2-D HG and LG-CH expansions are local with an effective convergence radius increasing with the truncation order (say *M*).
- High precision reconstrunction of 2-D regions from multiple, expansions around adjacent points (frame theory).



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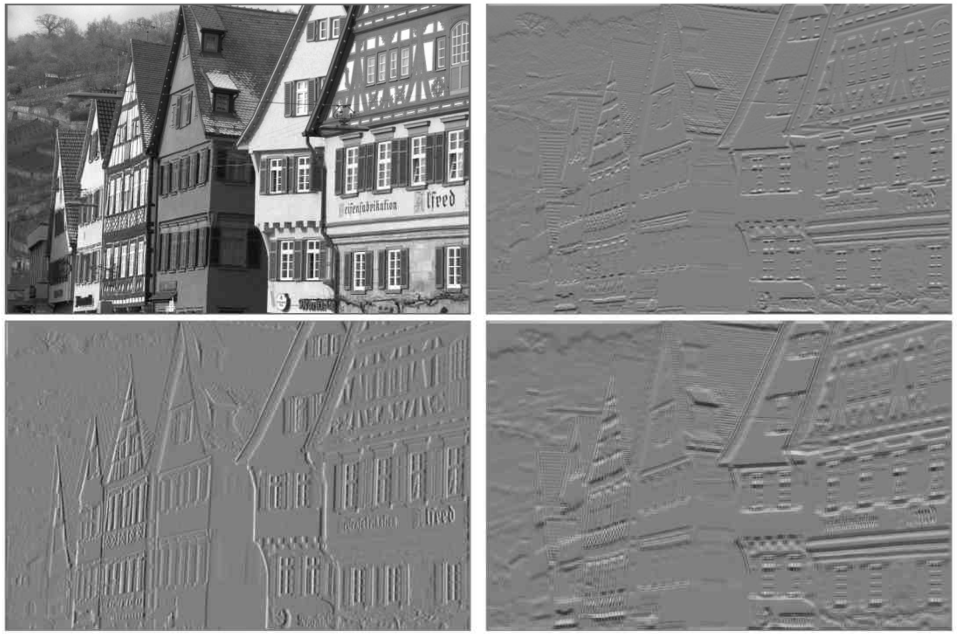
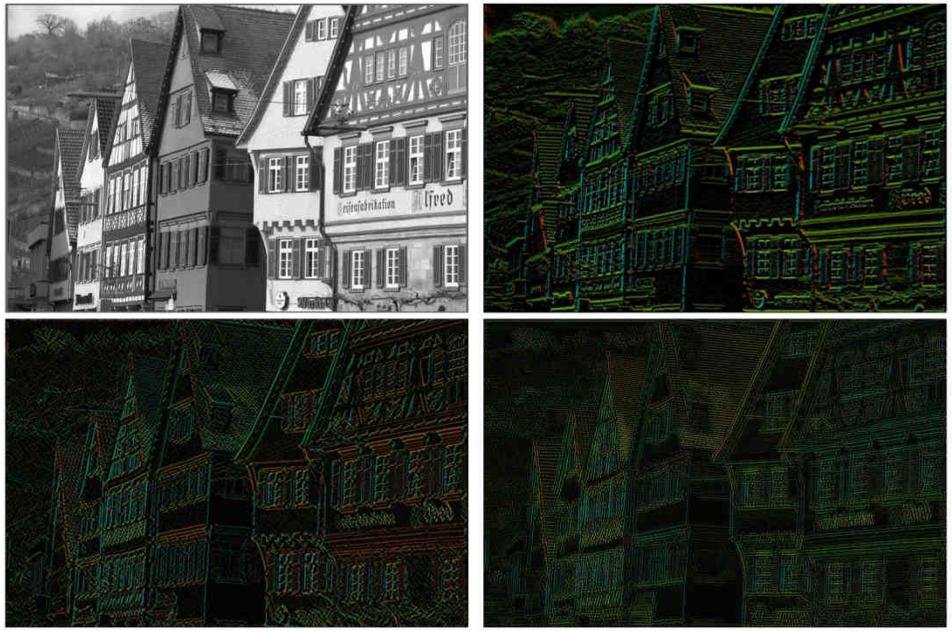
2-D HG and LG-CH image expansion examples

•2D-HG expansion



•LG-CH expansion (mag./phase)



Comparison between expansions

2-D HG expansion

- Derivative based (i.e., Taylor-type)
- Orthogonal
- Real valued functions and coefficients
- Cartesian separable (fast computation)
- Shows horizontal and vertical features

LG-CH expansion

- Circular harmonics based (i.e., Fourier-type)
- Orthogonal
- Complex valued functions and coefficients
- Polar separable (easily steerable)
- Shows complex features (edge, lines, crosses, etc…)

Main applications

- Pattern recognition and template matching;
- Feature extraction and parametric estimation;
- Signal and image enhancement and restoration;
- Optics:
  - point spread function analysis and deblurring;
  - paraxial approximation;
- Image rotation/stretching (LG-CH);
- Coding and compression;
- *Array processing* from antennas, microphones…

A challenging problem: *optimal* estimation of linear patterns and UWB plane wavefronts

- Linear patterns (edge, lines, bars, etc…) in image processing:
  - Feature classification;
  - Reconstruction and interpolation;
  - Enhancement (deblur and denoising).
- Ultra wide-band arrays for communications and remote sensing:
  - Strong model change with frequency;
  - Complex and inaccurate numerical representation in time or frequency domains;
  - Transient signals.
- But narrow-band array signal have a well-known, simple and accurate *subspace* representation…

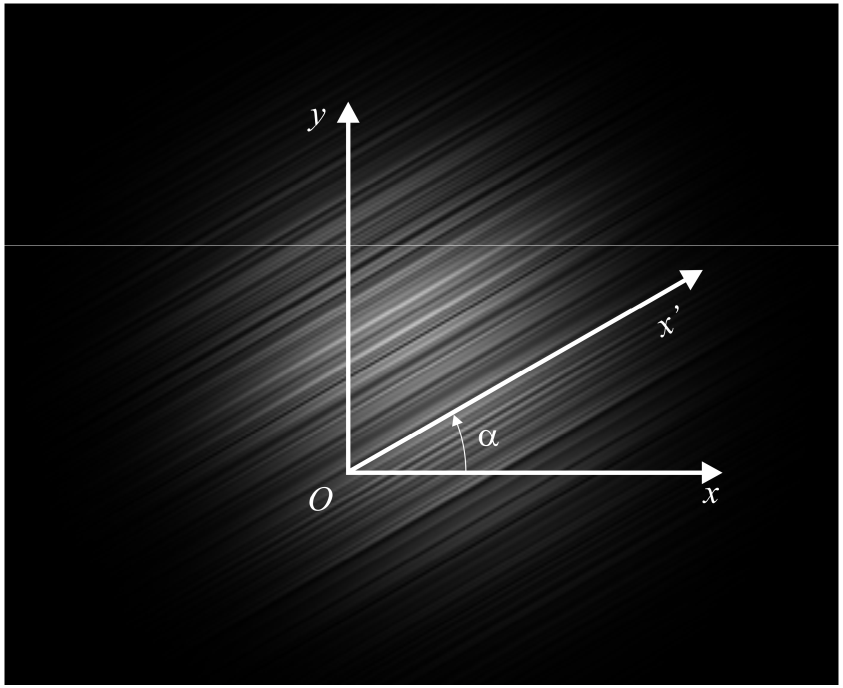
Existing Approaches (I)

- Time Delay of Arrival (TDOA) analysis:
  - Consistent single source localization capability only;
  - Difficult handling of correlated background noise and interference;
  - Estimation outliers at low SNR;
  - Sensitivity to array mis-calibration.
- Wide-band extension of narrowband parametric array techniques (ML, MUSIC, WSF,…) via frequency binning:
  - Difficult and costly numerical optimization;
  - Assume stationary signals and noise (sub-optimality with respect to transient signals);
  - Bias and excess estimation variance due to finite bandwidth and cross-over effects within each frequency bin.

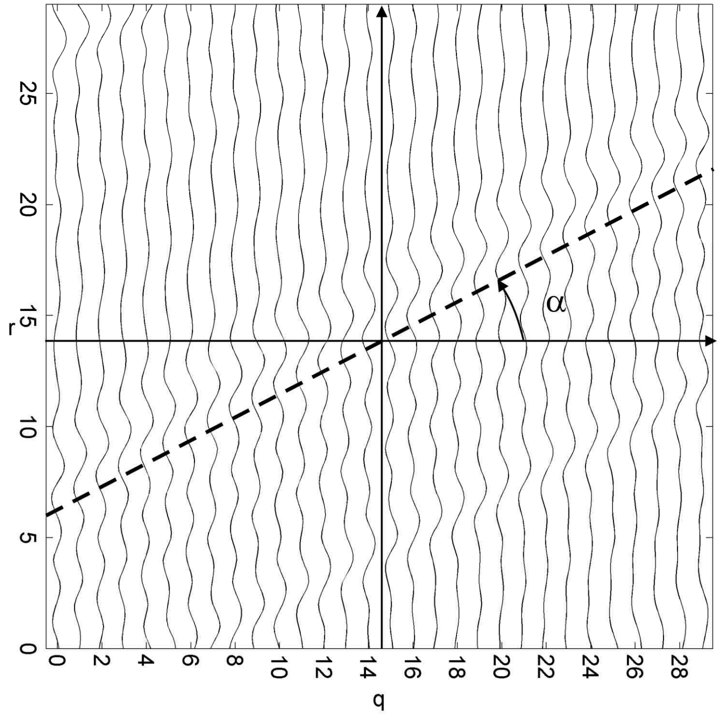
Existing approaches (II)

- Focusing and steering techniques:
  - Require binning or large multi-channel convolutions;
  - Use reduced statistics for non-white noise and/or non-unitary transformations: generally suboptimal estimates;
  - Modeling accuracy steeply decreases for bandwidths spanning more than one octave, introducing bias and excess estimation variance.
- **A more accurate mapping of the UWB Uniform Linear Array (ULA) response is sought, essentially insensitive to the signal bandwidth.**

Analogy between ULA signals in space-time and 1-D linear patterns in 2-D images



Time



Linear array sensor numbering

Space-time ULA UWB baseband signal model

2-D space-time patch

$$u(x,t) = s\left[t - \sin(\theta)\frac{x}{c}\right]$$

Linear pattern equation

$$u(q,r) = s\left\{\frac{T}{\cos(\alpha)}\left[r\cos(\alpha) - q\sin(\alpha)\right]\right\}$$

$$\tan(\alpha) = d\sin(\theta)$$

Moderate over-sampling

$$T < \frac{1}{2 f_{\max}\sqrt{1+d^2}}; \quad d = \frac{D}{cT}$$

Physical inter-sensor spacing

Wave propagation speed

Laguerre signal subspace

- The LG-CH coefficient vector of a linear wave-front generates a *signal subspace*:

$$\mathbf{f}_M = \mathbf{T}^H \mathbf{D}(-\alpha)\mathbf{E}_s\mathbf{c} + \mathbf{n}$$

$$\mathbf{g}_M = \mathbf{D}(-\alpha)\mathbf{E}_s\mathbf{c} + \mathbf{v}$$

$$c_m = \left[\sum_{h=0}^{M-m} \frac{(-1)^h \sigma\sqrt{\pi} H_h^2(0)}{2^{h-1} h!}\right]^{-\frac{1}{2}} b_m$$

$$b_m = \int_{-\infty}^{+\infty} s\left[\frac{t}{\cos(\alpha)}\right] \frac{H_m\left(t/\sigma\right)}{\sqrt{2^m m! \sigma\sqrt{\pi}}} e^{-\frac{t^2}{2\sigma^2}} dt$$

2-D Hermite model

LG-CH model

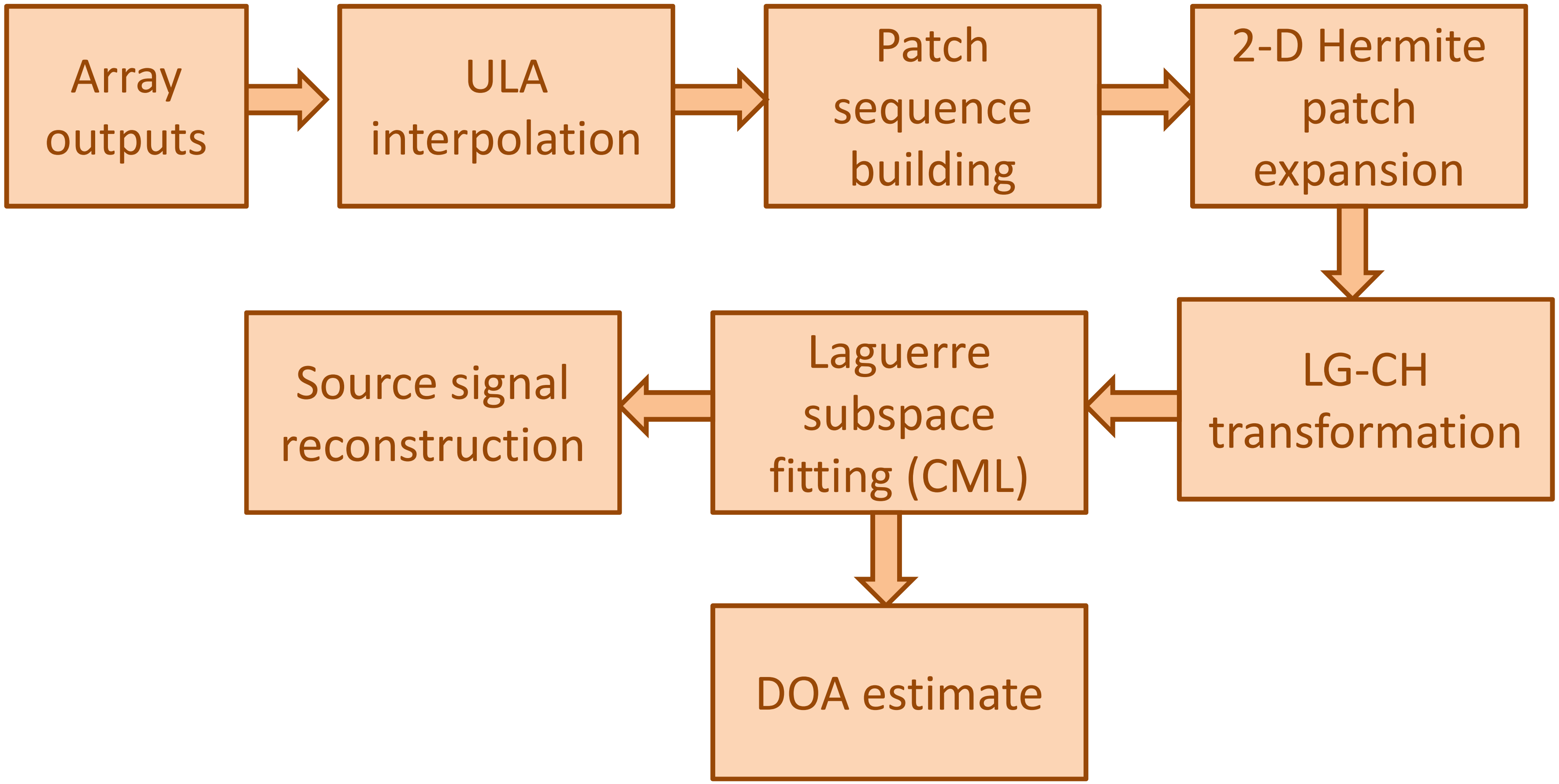
Real tomographic (array) gain

LG-CH diagonal (phase) steering matrix

Laguerre subspace

1-D Hermite real expansion of α-stretched signal

Basic LG-CH CML array processing



2-D Hermite-Gauss expansion

- The array signal is broken into consecutive patches, even partially overlapped.
- The coefficients of a truncated *M*-th order 2-D HG expansion of each patch are found by weighted LS fitting (Gaussian MLE).

Noise covariance inverse square root

2-D HG sampled functions

$$\mathbf{C}_\eta^{-1}[\mathbf{H}]\mathbf{f}_M \cong \mathbf{C}_\eta^{-1}\begin{bmatrix}\mathbf{u}(:,1)\\ \mathbf{u}(:,2)\\ \vdots\\ \mathbf{u}(;,R)\end{bmatrix}$$

Patch of array samples

Finite patch support issues

- Upper expansion order *M* is limited.
  - Possible bias from truncation (slight oversampling required);
  - Possible ill-conditioning (use regularization).
- Beamspace transformation: Fisher information loss.
  - Expansion radius should entirely cover the patch.
  - Use partially overlapped patches and related sufficient statistics.
  - Implicit space-time extrapolation from LS fitting.
- Aliasing of time-sampled functions.
  - Band-limited and space-time truncated versions of 2-D Hermite and LG-CH functions retain all relevant properties in UWB applications.

LG-CH CML estimation

- The conditional ML estimator is easily derived in the LG-CH domain for a single source in correlated noise and interference (WSF type optimization).

Coefficient covariance inverse square root

Rational cost function in $e^{j\alpha}$

Beamforming formula

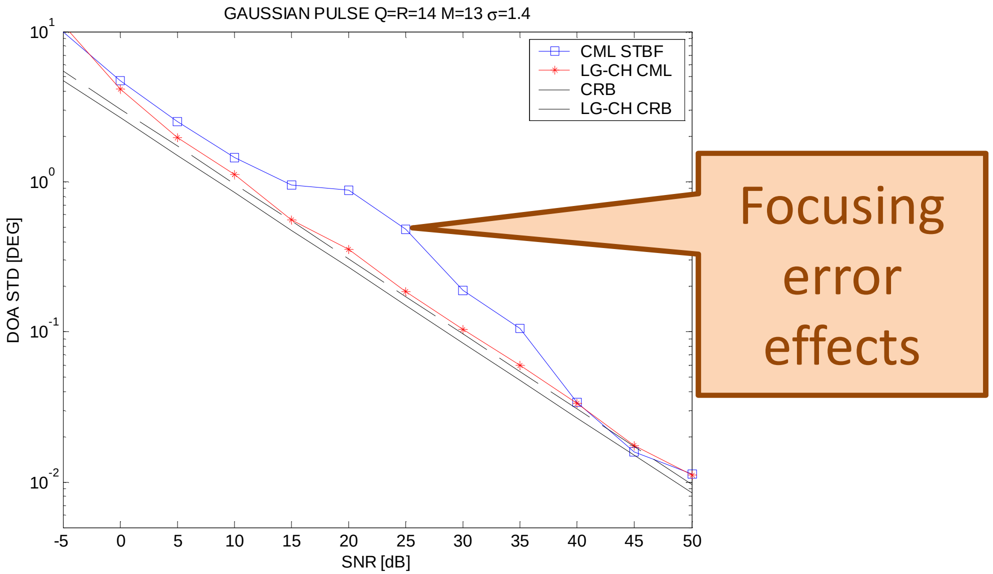
$$\hat{\alpha}_{CML} = \arg\min_{\alpha,\mathbf{c}(\alpha)} \left\| \mathbf{C}_v^{-1} \left[\mathbf{g}_M - \mathbf{D}(-\alpha)\mathbf{E}_s\mathbf{c}(\alpha) \right] \right\|_2^2$$

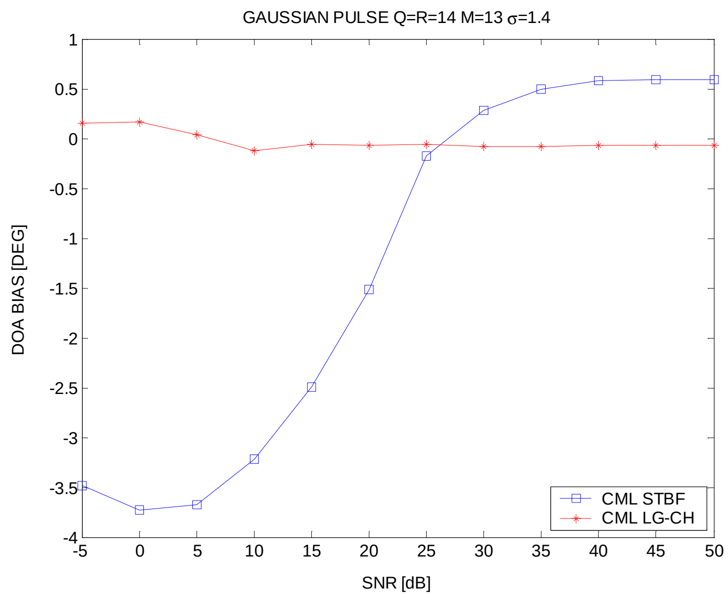
$$\mathbf{c}(\alpha) = \left[\mathbf{E}_s^H \mathbf{D}(\alpha) \mathbf{R}_{vv}^{-1} \mathbf{D}(-\alpha) \mathbf{E}_s \right]^{-1} \mathbf{E}_s^H \mathbf{D}(\alpha) \mathbf{R}_{vv}^{-1} \mathbf{g}$$

$$\hat{\mathbf{c}}_{CML} = \mathbf{c}(\hat{\alpha}_{CML}); \quad \hat{\theta}_{CML} = \arcsin\left[\frac{\tan(\hat{\alpha}_{CML})}{d} \right]$$

Short UWB Gaussian pulse signal

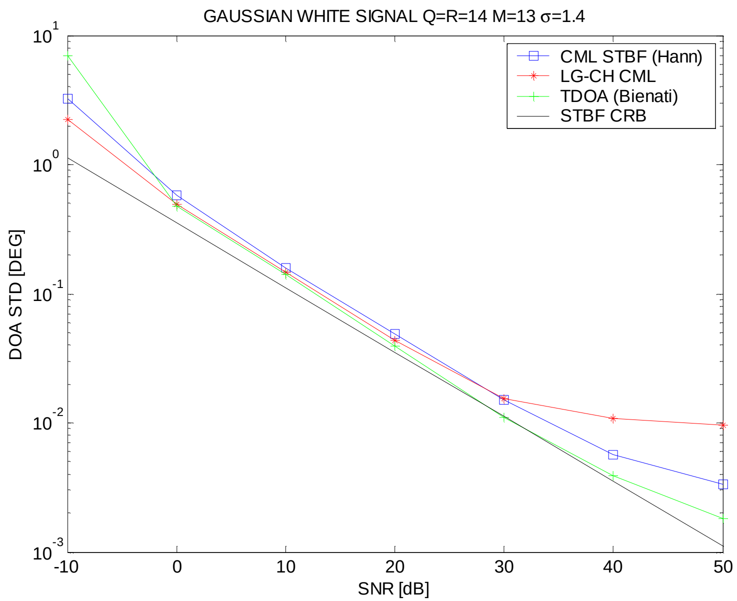
- DOA 35°; AWGN; LG-CH CML *M*=13; σ=1.4; *Q*=*R*=14; single patch (i.e., UWB monopulse).
- TDOA does not work: comparison with CML steered beamformer (rectangular window) only.

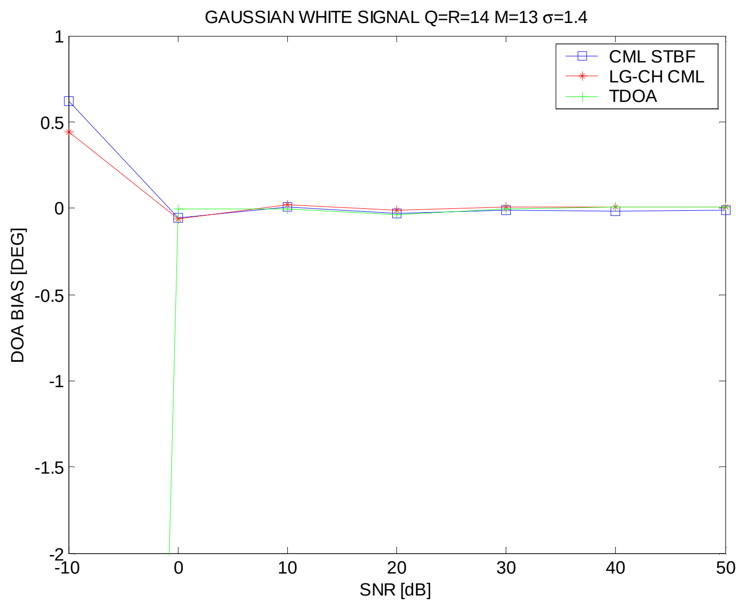




Random white signal

- DOA 35°; AWGN; LG-CH CML M=13; σ=1.4; Q=R=14; 140 time samples; overlap=11.





Conclusions and future work

- The LG-CH CML has low estimation bias and variance close to the beamspace CRB in the low-to-mid SNR region.
- Computationally efficient CML on a PC machine (x4 speed w.r.t. STBF and x20 w.r.t. TDOA on MATLAB®).
- Extensions to other subspace algorithms (e.g., MUSIC) of the LG-CH mapping.
- Extension to multi-source environments.
- Improved expansion coefficient estimation.

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